Various Logical Consequences in Modal Logic

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Basics of Modal Logic

- Basic modal language: $\mathcal{L}_\Box \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$, where $p \in PV$
- (Relational) Frame: $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and $R \subseteq W \times W$
- (Relational) Model: $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame, and $V : PV \to \wp(W)$ is the valuation function
- Truth conditions: Given a model $\mathfrak{M} = (W, R, V)$ for $\mathcal{L}_\Box$, $\mathfrak{M}, w \Vdash \varphi$ is defined as follows.
  - $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$
  - $\mathfrak{M}, w \Vdash \varphi \wedge \psi$ iff $\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$
  - $\mathfrak{M}, w \Vdash \Box\varphi$ iff for all $u \in R(w)$, $\mathfrak{M}, u \Vdash \varphi$

Basics of Modal Logic

- Submodels / Subframes
- Generated submodels /subframes
- Typical Class of Frames: K, D, T, B, K4, K45, S4, S5, U
- Typical Axiomatic Systems: $\mathbf{K}$, $\mathbf{D}$, $\mathbf{T}$, $\mathbf{B}$, $\mathbf{K4}$, $\mathbf{K45}$, $\mathbf{S4}$, $\mathbf{S5}$

Notations

- $\mathfrak{M}, w \Vdash \Gamma$ means $\mathfrak{M}, w \Vdash \varphi$ for all $\varphi \in \Gamma$.
- $\mathfrak{M} \Vdash \varphi$ means $\mathfrak{M}, w \Vdash \varphi$ for all w in $\mathfrak{M}$.
- $\Box^0\varphi := \varphi$, $\Box^{n+1}\varphi := \Box\Box^n\varphi$
- $\Box_r\varphi := \varphi \wedge \Box\varphi$
- $\Box\Gamma := \{\Box\varphi \mid \varphi \in \Gamma\}$
- $\Box_r\Gamma := \{\Box_r\varphi \mid \varphi \in \Gamma\}$
- $\Box^\omega\Gamma := \{\Box^n\varphi \mid n \in \mathbb{N}, \varphi \in \Gamma\}$
- $\Box^\omega\varphi := \Box^\omega\{\varphi\}$
- R^*: the reflexive and transitive closure of R
- $\mathcal{L}_0$: the set of modal-free formulas in $\mathcal{L}_\Box$.

Question

- How many logical consequences do we need in modal logic?
- Probably one!

Outline

1. Global Consequence
   - Global semantic consequence
   - Global syntactic consequence
2. Kripke Consequence
   - Basic modal language
   - Adding the actuality operator
3. Informational Consequence
   - Basic modal language
   - Adding indicative conditionals
4. Tenary Consequence
   - Fitting consequence
   - Update consequence
5. A Uniform Framework for Logical Consequence
   - Semantic consequence
   - Syntactic consequence
   - A more general semantic consequence

Outline

Local Consequence vs. Global Consequence

- Let F be a class of frames.

Definition 1 (Local consequence)

φ is a *local consequence* of Γ with respect to F, denoted $\Gamma \vDash_{\mathsf{F}} \varphi$, if for all frames $\mathfrak{F}$ in F, for all models $\mathfrak{M}$ based on $\mathfrak{F}$, for all w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \Gamma$ implies $\mathfrak{M}, w \Vdash \varphi$.

Definition 2 (Global consequence)

φ is a *global consequence* of Γ with respect to F, denote $\Gamma \vDash^{g}_{\mathsf{F}} \varphi$, if for all frames $\mathfrak{F}$ in F, for all models $\mathfrak{M}$ based on $\mathfrak{F}$, $\mathfrak{M} \Vdash \Gamma$ implies $\mathfrak{M} \Vdash \varphi$.

Example 3

1. $p \nvDash_{\mathsf{K}} \Box p$
2. $p \vDash^{g}_{\mathsf{K}} \Box p$

Local Consequence vs. Global Consequence

Fact 4

For any class of frames F, *for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$,

1. $\vDash^g_{\mathsf{F}} \varphi$ *iff* $\vDash_{\mathsf{F}} \varphi$;
2. $\Gamma \vDash_{\mathsf{F}} \varphi$ *implies* $\Gamma \vDash^g_{\mathsf{F}} \varphi$

Questions

1. Can global consequence be defined by local consequence?
2. Can local consequence be defined by global consequence?
3. What's the deduction theorem for global consequence?

Defining Global Consequence by Local Consequence

Lemma 5

For any frame $\mathfrak{F}$, *for any* w *and* w' *in* $\mathfrak{F}$, *let* V *and* V' *be valuations on* $\mathfrak{F}$ *such that for all* $p \in PV$, $w \in V(p)$ *iff* $w' \in V(p)$. *Then for all* $\varphi \in \mathcal{L}_0$, $\mathfrak{F}, V, w \Vdash \varphi$ *iff* $\mathfrak{F}, V', w' \Vdash \varphi$.

Fact 6

Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_0$. *Then for any class of frames* F, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Gamma \vDash_{\mathsf{F}} \varphi$.

Fact 7

Let $\Gamma \subseteq \mathcal{L}_0$ *and* $\varphi \in \mathcal{L}_\Box$. *Then for any class of frames* F, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box^\omega \Gamma \vDash_{\mathsf{F}} \varphi$.

Defining Global Consequence by Local Consequence

Definition 8

Given a relational model $\mathfrak{M} = (W, R, V)$, define the operator $\boxplus$ as follows:

$$\mathfrak{M}, w \Vdash \boxplus\varphi \text{ iff for all } u \in R^*(w), \mathfrak{M}, u \Vdash \varphi.$$

- $\mathcal{L}_{\Box\boxplus} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \boxplus\varphi$

Fact 9 (Venema, 1992, p. 159)

For any class of frames F*, for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box\boxplus}$, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\boxplus\Gamma \vDash_{\mathsf{F}} \boxplus\varphi$.

Defining Global Consequence by Local Consequence

Definition 10

Given a relational model $\mathfrak{M} = (W, R, V)$, define the *universal modality* U as follows,

$$\mathfrak{M}, w \Vdash U\varphi \text{ iff for all } u \in W, \mathfrak{M}, u \Vdash \varphi.$$

- $\mathcal{L}_{\Box U} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid U\varphi$

Fact 11 (Goranko and Passy, 1992)

For any class of frames F, *for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box U}$, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $U\Gamma \vDash_{\mathsf{F}} \varphi$ *iff* $U\Gamma \vDash_{\mathsf{F}} U\varphi$

Defining Global Consequence by Local Consequence

Question

Can global consequence be defined by local consequence within the basic modal language?

Defining Global Consequence by Local Consequence

Fact 12

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash^g_{\mathsf{K}} \varphi$ *iff* $\Box^\omega \Gamma \vDash_{\mathsf{K}} \varphi$.

The fact is an exercise in Blackburn et al. (2001, p. 32). We generalize it as follows.

Theorem 13

Let F *be any class of frames that is closed under point generated subframes. Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box^\omega \Gamma \vDash_{\mathsf{F}} \varphi$.

The direction from right to left does not require F to be closed under point generated subframes. The condition of closure under point generated subframes can not be dropped for the other direction.

Defining Global Consequence by Local Consequence

Fact 14

There exist a class of frames F *and formulas* $\Gamma \cup \{\varphi\}$ *such that* $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *but* $\Box^\omega \Gamma \nvDash_{\mathsf{F}} \varphi$.

Proof.

Let F be a singleton frame $\mathfrak{F} = (\{w, u\}, (w, u))$. Then for any valuation V on $\mathfrak{F}$, $\mathfrak{F}, V \nVdash \Box\bot$, since $\mathfrak{F}, V, w \nVdash \Box\bot$. Hence, $\Box\bot \vDash^g_{\mathsf{F}} \bot$. On the other hand, given any valuation V on $\mathfrak{F}$, $\mathfrak{F}, V, u \Vdash \Box^n\Box\bot$ for all $n \in \mathbb{N}$, but $\mathfrak{F}, V, u \nVdash \bot$. Hence, $\Box^\omega\Box\bot \nvDash_{\mathsf{F}} \bot$. □

de Rijke and Wansing (2006, p. 425) claim that the equivalence between $\Gamma \vDash^g_{\mathsf{F}} \varphi$ and $\Box^\omega\Gamma \vDash_{\mathsf{F}} \varphi$ holds for all F, which is incorrect by the above fact.

Defining Global Consequence by Local Consequence

Corollary 15

Let F *be any class of transitive frames that is closed under point generated subframes. Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box_r\Gamma \vDash_{\mathsf{F}} \varphi$.

Question

Can we simplify further?

Defining Global Consequence by Local Consequence

Definition 16

A class of frames F is *closed under irreflexive point extension*, if for any frame $\mathfrak{F} = (W, R)$ in F, for any $w \in W$ with $\neg Rxx$, any point extension $\mathfrak{F}' = (W', R')$ of $\mathfrak{F}$ for w by $u \notin W$ is also in F, where $\mathfrak{F}'$ is defined as follows.

$$W' = W \cup \{u\}$$
$$R' = R \cup \{(u, w)\} \cup \{(u, w') \mid (w, w') \in R\}$$

Theorem 17

Let F *be any class of transitive frames that is closed under point generated subframes and irreflexive point extension. Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{F}} \Box\varphi$.

Defining Global Consequence by Local Consequence

Corollary 18

Let F *be any class of reflexive and transitive frames that is closed under point generated subframes. Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$*,* $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{F}} \Box\varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{F}} \varphi$*.*

Corollary 19

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$*, for any* F *in* $\{\mathsf{K4}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$*,* $\Gamma \vDash^g_{\mathsf{F}} \varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{F}} \Box\varphi$*.*

Defining Local Consequence by Global Consequence

Definition 20

Given a model $\mathfrak{M}$, define the 'only' operator as follows.

$$\mathfrak{M}, w \Vdash O\varphi \text{ iff } \mathfrak{M}, w \Vdash \varphi \text{ and for all } w' \neq w, \mathfrak{M}, w' \nVdash \varphi.$$

- $\mathcal{L}_{\Box UO} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid U\varphi \mid O\varphi$

Theorem 21 (Van Benthem)

For any class of frames F, *for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box UO}$, *for any* $p \notin Var(\Gamma \cup \{\varphi\})$

$$\Gamma \vDash_{\mathsf{F}} \varphi \textit{ iff } \{EOp\} \cup \{p \to \gamma \mid \gamma \in \Gamma\} \vDash^g_{\mathsf{F}} p \to \varphi,$$

where E *is the dual of* U, *namely,* $E\varphi = \neg U \neg\varphi$.

Deduction Theorem

Fact 22 (Blackburn et al., 2001, p. 32)

For any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma, \varphi \vDash^g_{\mathsf{S4}} \psi$ *iff* $\Gamma \vDash^g_{\mathsf{S4}} \Box\varphi \to \psi$.

We generalize it as follows.

Theorem 23

Let F *be any class of transitive frames that is closed under point generated subframes . Then for any* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma, \varphi \vDash^g_{\mathsf{F}} \psi$ *iff* $\Gamma \vDash^g_{\mathsf{F}} \Box_r\varphi \to \psi$.

Corollary 24

For any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma, \varphi \vDash^g_{\mathsf{K4}} \psi$ *iff* $\Gamma \vDash^g_{\mathsf{K4}} \Box_r\varphi \to \psi$.

Outline

Lobal Consequence vs. Global Consequence

- Let $\mathbf{S}$ be an axiomatic system.

Definition 25 (Local consequence)

φ is a *local consequence* of Γ in $\mathbf{S}$, denoted $\Gamma \vdash_{\mathbf{S}} \varphi$, iff there exists a finite subset Δ of Γ such that $\vdash_{\mathbf{S}} \bigwedge \Delta \to \varphi$, i.e., there is a proof of $\bigwedge \Delta \to \varphi$, using axioms and rules of $\mathbf{S}$.

Definition 26 (Global consequence)

φ is a *global consequence* of Γ in $\mathbf{S}$, denoted $\Gamma \vdash^{g}_{\mathbf{S}} \varphi$, iff there is a derivation from Γ to φ, using axioms and rules of $\mathbf{S}$.

Defining Global Consequence by Local Consequence

Theorem 27

Let $\mathbf{S}$ *be any axiomatic extension of* $\mathbf{K}$. *Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vdash^g_{\mathbf{S}} \varphi$ *iff* $\Box^\omega \Gamma \vdash_{\mathbf{S}} \varphi$.

Corollary 28

Let $\mathbf{S}$ *be any axiomatic extension of* $\mathbf{K4}$. *Then for any* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vdash^g_{\mathbf{S}} \varphi$ *iff* $\Box_r \Gamma \vdash_{\mathbf{S}} \varphi$.

Corollary 29

Let $\mathbf{S}$ *be any axiomatic extension of* $\mathbf{S4}$. *Then for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vdash^g_{\mathbf{S}} \varphi$ *iff* $\Box\Gamma \vdash_{\mathbf{S}} \varphi$ *iff* $\Box\Gamma \vdash_{\mathbf{S}} \Box\varphi$.

Deduction Theorem

Theorem 30

Let $\mathbf{S}$ *be any axiomatic extension of* $\mathbf{K4}$*. Then for any* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma, \varphi \vdash^g_{\mathbf{S}} \psi$ *iff* $\Gamma \vdash^g_{\mathbf{S}} \Box_r \varphi \to \psi$.

Corollary 31

Let $\mathbf{S}$ *be any axiomatic extension of* $\mathbf{S4}$*. Then for any* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$, $\Gamma, \varphi \vdash^g_{\mathbf{S}} \psi$ *iff* $\Gamma \vdash^g_{\mathbf{S}} \Box \varphi \to \psi$.

Completeness

Lemma 32

Let $\mathbf{S}$ be any axiomatic system whose rules of inference include Necessitation. Then for any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_\Box$,

1. *if $\Gamma, \Box\varphi \vdash^g_{\mathbf{S}} \psi$ then $\Gamma, \varphi \vdash^g_{\mathbf{S}} \psi$;*
2. *if $\Box^\omega\Gamma \vdash^g_{\mathbf{S}} \varphi$ then $\Gamma \vdash^g_{\mathbf{S}} \varphi$.*

Theorem 33

Let $\mathbf{S}$ be the axiomatic system for any normal modal logic that is sound and strongly complete with respect to $\vDash_{\mathsf{F}}$. Then $\vdash^g_{\mathbf{S}}$ is sound and strongly complete with respect to $\vDash^g_{\mathsf{F}}$.

Outline

Kripke Frames

Definition 34

A *Kripke frame* is a triple $\mathfrak{F}^u = (W, R, u)$, where $\mathfrak{F} = (W, R)$ is a standard frame and $u \in W$ is called the *actual world* of the frame. Given a Kripke frame $\mathfrak{F}^u$ and a valuation $V : PV \to \wp(W)$ on $\mathfrak{F}$, call $(\mathfrak{F}^u, V)$ a *Kripke model based on* $\mathfrak{F}^u$.

Given a class of Kripke frames $\mathcal{F}$, let $\mathsf{F}(\mathcal{F}) = \{\mathfrak{F} \mid \mathfrak{F}^u \in \mathcal{F}\}$ be the class of (relational) frames underlying $\mathcal{F}$. It is easily seen that for the class of all Kripke frames $\mathcal{K}$, $\mathsf{F}(\mathcal{K}) = \mathsf{K}$ is just the class of all (relational) frames.

Truth Conditions in Kripke Frames

Definition 35

Given a Kripke frame $\mathfrak{F}^u = (W, R, u)$, a valuation V on $\mathfrak{F}$ and a world $w \in W$, that φ is true at w in $(\mathfrak{F}^u, V)$, denoted $\mathfrak{F}^u, V \Vdash_w \varphi$, is defined as follows.

- $\mathfrak{F}^u, V \Vdash_w p$ iff $w \in V(p)$,
- $\mathfrak{F}^u, V \Vdash_w \neg\varphi$ iff $\mathfrak{F}^u, V \nVdash_w \varphi$,
- $\mathfrak{F}^u, V \Vdash_w \varphi \wedge \psi$ iff $\mathfrak{F}^u, V \Vdash_w \varphi$ and $\mathfrak{F}^u, V \Vdash_w \psi$,
- $\mathfrak{F}^u, V \Vdash_w \Box\varphi$ iff for all $w' \in R(w)$, $\mathfrak{F}^u, V \Vdash_{w'} \varphi$.

We write $\mathfrak{F}^u, V \Vdash \varphi$ if for all w in $\mathfrak{F}$, $\mathfrak{F}^u, V \Vdash_w \varphi$.

Kripke Consequence

Definition 36 (Kripke consequence)

Given a class of Kripke frames $\mathcal{F}$, the *Kripke consequence* with respect to $\mathcal{F}$, denoted $\vDash^k_{\mathcal{F}}$, is defined as follows: $\Gamma \vDash^k_{\mathcal{F}} \varphi$ iff for all frames $\mathfrak{F}^u$ in $\mathcal{F}$, for all valuations V on $\mathfrak{F}$, $\mathfrak{F}^u, V \Vdash_u \Gamma$ implies $\mathfrak{F}^u, V \Vdash_u \varphi$.

Fact 37

For $\mathcal{L}_\Box$, there exists $\mathcal{F}$ such that $\vDash^k_{\mathcal{F}}$ and $\vDash_{\mathcal{F}}$ do not coincide.

Proof.

Consider $\mathfrak{F} = (\{w, u\}, \{(w, u)\})$ and $\mathcal{F} = \{\mathfrak{F}^w, \mathfrak{F}^u\}$. Then $\vDash^k_{\mathcal{F}} \Box\bot$ but $\nvDash_{\mathcal{F}} \Box\bot$. □

Question

Question

When do Kripke consequence and local consequence coincide?

Kripke Consequence

Definition 38

A class of Kripke frames $\mathcal{F}$ is *complete for actuality*, if for every $\mathfrak{F}^u = (W, R, u)$ in $\mathcal{F}$ and every $w \in W$, $\mathfrak{F}^w = (W, R, w)$ is also in $\mathcal{F}$.

Lemma 39

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, for any frame $\mathfrak{F} = (W, R)$, any valuation V on $\mathfrak{F}$ and any $w \in W$, $\mathfrak{F}, V, w \Vdash \varphi$ iff $\mathfrak{F}^w, V \Vdash_w \varphi$.

Theorem 40

For any class of Kripke frames $\mathcal{F}$ that is complete for actuality, for any $\Gamma \cup \varphi \subseteq \mathcal{L}_\Box$, $\Gamma \vDash_{\mathcal{F}} \varphi$ iff $\Gamma \vDash^k_{\mathcal{F}} \varphi$.

Kripke Consequence

Corollary 41

For any $\Gamma \cup \varphi \subseteq \mathcal{L}_\Box$, $\Gamma \vDash_{\mathcal{K}} \varphi$ *iff* $\Gamma \vDash^k_{\mathsf{K}} \varphi$.

Given a class of standard frames F, let $\mathcal{F}(\mathsf{F}) = \{\mathfrak{F}^w \mid \mathfrak{F} = (W, R) \in \mathsf{F}, w \in W\}$ be the class of Kripke frames generated by F. It is easily seen that $\mathcal{F}(\mathsf{F})$ is complete for actuality. Then Theorem 40 can be reformulated as follows.

Theorem 42

For any class of (relational) frames F, *for any* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash_{\mathsf{F}} \varphi$ *iff* $\Gamma \vDash^k_{\mathcal{F}(\mathsf{F})} \varphi$.

Outline

Adding the Actuality Operator

Definition 43 (Actuality operator)

Given a Kripke model $(\mathfrak{F}^u, V)$, define the *actuality operator* $@$ as follows.

$$\mathfrak{F}^u, V \Vdash_w @\varphi \text{ iff } \mathfrak{F}^u, V \Vdash_u \varphi.$$

- $\mathcal{L}_{\Box @} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid @\varphi$

Fact 44

For $\mathcal{L}_{\Box @}$, $\vDash^k_{\mathcal{F}}$ and $\vDash_{\mathcal{F}}$ do not coincide, even if $\mathcal{F}$ is complete for actuality.

Proof.

Consider $\mathfrak{F} = (\{w, u\}, \emptyset)$ and $\mathcal{F} = \{\mathfrak{F}^w\}$. Let $V(p) = \{u\}$. Then $p \vDash^k_{\mathcal{F}} @p$, but $p \nvDash_{\mathcal{F}} @p$. $\Box$

Defining Kripke Consequence by Local and Global Consequence

Theorem 45

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box @}$, *for any class of Kripke frames* $\mathcal{F}$, $\Gamma \models^k_{\mathcal{F}} \varphi$ *iff* $@\Gamma \models^k_{\mathcal{F}} @\varphi$ *iff* $@\Gamma \models_{\mathcal{F}} @\varphi$ *iff* $@\Gamma \models^g_{\mathcal{F}} @\varphi$.

Defining Local Validity by Kripke Validity

- $\mathcal{L}_{\Box @ U} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid @\varphi \mid U\varphi$

Fact 46 (Humberstone (2004))

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box @ U}$, *for any class of Kripke frames* $\mathcal{F}$, $\vDash_{\mathcal{F}} \varphi$ *iff* $\vDash^{k}_{\mathcal{F}} U\varphi$.

Defining Local Consequence by Global Consequence Revisited

Theorem 47

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, *for any class of frames* F, $\Gamma \vDash_{\mathsf{F}} \varphi$ *iff* $@\Gamma \vDash^g_{\mathcal{F}(\mathsf{F})} @\varphi$.

Outline

Motivation

Yalcin (2007) advocated a non-classical consequence relation, called informational consequence, motivated by the following desiderata.

- Consequence is classical: $\vDash$ respects classical entailment patterns.
- Nonfactivity of epistemic possibility: $\Diamond\varphi \nvDash \varphi$
- Epistemic contradiction: $\neg\varphi \wedge \Diamond\varphi \vDash \bot$

Domain Semantics

Definition 48

A *domain model* is a pair $\mathfrak{D} = (W, V)$, where $W \neq \emptyset$ and $V : PV \to \wp(W)$ is a valuation function.

Definition 49 (Truth conditions)

Given a domain model $\mathfrak{D} = (W, V)$, that φ is true at $(w, i) \in W \times \wp(W)$ in $\mathfrak{D}$, denoted $\mathfrak{D}, w, i \Vdash \varphi$, is inductively defined as follows, where $\mathfrak{D}, i \Vdash \varphi$ means for all $w \in i$, $\mathfrak{D}, w, i \Vdash \varphi$.

- $\mathfrak{D}, w, i \Vdash p$ iff $w \in V(p)$
- $\mathfrak{D}, w, i \Vdash \neg\varphi$ iff $\mathfrak{D}, w, i \nVdash \varphi$
- $\mathfrak{D}, w, i \Vdash \varphi \wedge \psi$ iff $\mathfrak{D}, w, i \Vdash \varphi$ and $\mathfrak{D}, w, i \Vdash \psi$
- $\mathfrak{D}, w, i \Vdash \Box\varphi$ iff $\mathfrak{D}, i \Vdash \varphi$

Relating Informational and Global Consequence

Definition 50 (Informational consequence)

φ is an *informational consequence* of Γ, denoted $\Gamma \vDash_I \varphi$, if for all domain models $\mathfrak{D} = (W, V)$ and $i \subseteq W$, $\mathfrak{D}, i \Vdash \Gamma$ implies $\mathfrak{D}, i \Vdash \varphi$.

Theorem 51 (Schulz (2010))

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash_I \varphi$ *iff* $\Gamma \vDash^g_{\mathsf{S5}} \varphi$

Outline

Truth Conditions for Indicative Conditionals

- $\mathcal{L}_{\Box\Rightarrow} \ni \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid (\psi \Rightarrow \varphi)$, where $\psi \in \mathcal{L}_0$

Definition 52

Given a domain model $\mathfrak{D} = (W, V)$, the indicative operator $\Rightarrow$ is defined as follows.

$$\mathfrak{D}, w, i \Vdash \varphi \Rightarrow \psi \text{ iff } \mathfrak{D}, i + \varphi \Vdash \psi,$$

where $i + \varphi$ is the maximal $i' \subseteq i$ such that $\mathfrak{D}, i' \Vdash \varphi$.

- Since φ in $\varphi \Rightarrow \psi$ does not contain modalities, $i + \varphi$ is just $\{w' \in i \mid \mathfrak{D}, w', i \Vdash \varphi\}$.

Lemma 53

For any domain model $\mathfrak{D} = (W, V)$, any $w \in W$ and $i \subseteq W$, $\mathfrak{D}, w, i \Vdash \varphi$ iff $\mathfrak{D}_i, w, i \Vdash \varphi$, where $\mathfrak{D}_i = (i, V\restriction_i)$.

Truth Conditions for Indicative Conditionals

Fact 54

For any $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_{\Box\Rightarrow}$,

1. $\varphi \Rightarrow \psi, \varphi \vDash_I \psi$
2. $\varphi \Rightarrow \psi, \neg\psi \nvDash_I \neg\varphi$
3. $\varphi \Rightarrow \Box\psi, \neg\psi \vDash_I \neg\varphi$
4. $\varphi \Rightarrow \psi, \neg\psi \vDash_I \Diamond\neg\varphi$
5. $\Gamma, \varphi \vDash_I \bot \not\Longleftrightarrow \Gamma \vDash_I \neg\varphi$
6. $\Gamma, \varphi \vee \psi, \varphi \Rightarrow \chi, \psi \Rightarrow \chi \nvDash_I \chi$
7. $\Gamma, \Box\varphi \vee \Box\psi, \varphi \Rightarrow \chi, \psi \Rightarrow \chi \vDash_I \chi$

Reducing Indicative Conditionals

Definition 55 (Holliday and Icard III 2017)

Define the translation $t : \mathcal{L}_{\Box\Rightarrow} \to \mathcal{L}_{PAL}$ as follows:

- $t(p) = p$, $p \in PV$
- $t(\neg\varphi) = \neg t(\varphi)$
- $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$
- $t(\Box\varphi) = \Box t(\varphi)$
- $t(\varphi \Rightarrow \psi) = [t(\varphi)]\Box t(\psi)$

Relating Informational and Global Consequence

Lemma 56

For any $\mathfrak{D} = (W, V)$, *for any* $w \in W$ *and* $i \subseteq W$ *with* $w \in i$, $\mathfrak{D}, w, i \Vdash \varphi$ *iff* $\mathfrak{D}_i, w \Vdash t(\varphi)$, *where* $\mathfrak{D}_i = (i, i \times i, V \restriction_i)$.

Lemma 57

For any $\mathfrak{M} = (W, R, V)$ *with* $R = W \times W$, $\mathfrak{M}, w \Vdash t(\varphi)$ *iff* $\mathfrak{D}^{\mathfrak{M}}, w, W \Vdash \varphi$, *where* $\mathfrak{D}^{\mathfrak{M}} = (W, V)$.

Lemma 58

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{PAL}$, $\Gamma \vDash^g_{\mathsf{U}} \varphi$ *iff* $\Gamma \vDash^g_{\mathsf{S5}} \varphi$.

Theorem 59

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Box\Rightarrow}$, $\Gamma \vDash_I \varphi$ *iff* $t(\Gamma) \vDash^g_{\mathsf{S5}} t(\varphi)$.

Relating Informational and Global Consequence

Theorem 60

Let $\varphi \Rightarrow \psi$ *denote* $[\varphi]\Box\psi$, *then for any* $\varphi, \psi \in \mathcal{L}_{PAL}$,

1. $\varphi, \varphi \Rightarrow \psi \not\models \psi$
2. $\varphi, \varphi \Rightarrow \psi \models^g \psi$
3. $\varphi \vee \psi, \varphi \Rightarrow \chi, \psi \Rightarrow \chi \not\models \chi$
4. $\varphi \vee \psi, \varphi \Rightarrow \chi, \psi \Rightarrow \chi \not\models^g \chi$

Outline

Semantic Fitting Consequence

Definition 61 (Fitting, 1983; Fitting and Mendelsohn, 1998, Def. 1.9.1)

Given a class of frames F, φ is a *(semantic) consequence of the global assumptions* Γ *and the local assumptions* Λ *in* F, denoted $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi$, if for every frame $\mathfrak{F}$ in F, for every model $\mathfrak{M}$ based on $\mathfrak{F}$ in which $\mathfrak{M} \Vdash \Gamma$, for every world w in $\mathfrak{M}$ at which $\mathfrak{M}, w \Vdash \Lambda$, we have $\mathfrak{M}, w \Vdash \varphi$.

Deduction Theorem

Theorem 62 (Deduction theorem, Fitting, 1983; Fitting and Mendelsohn, 1998, Prop. 1.9.6)

For any class of frames F *in* $\{\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{B}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}\}$ *and any* $\Gamma \cup \Lambda \cup \{\varphi, \psi\} \subseteq \mathcal{L}_{\Box}$,

1. $(\Gamma)\Lambda, \varphi \vDash_{\mathsf{F}} \psi$ *iff* $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi \to \psi$;
2. $(\Gamma, \varphi)\Lambda \vDash_{\mathsf{F}} \psi$ *iff* $(\Gamma)\Lambda \cup \Box^{\omega}\varphi \vDash_{\mathsf{F}} \psi$.

The second item of Theorem 62 is a corollary of Theorem 13.

Syntactic Fitting Consequence

Definition 63 (Fitting and Mendelsohn, 1998, Def. 3.3.1)

Given an axiomatic system $\mathbf{S}$ for $\mathcal{L}_{\Box}$, φ is a *(syntactic) consequence from the global assumptions* Γ *and the local assumptions* Λ *in* $\mathbf{S}$, denote $(\Gamma)\Lambda \vdash_{\mathbf{S}} \varphi$, if there is a sequence of formulas satisfying the following conditions.

1. The sequence is divided into two separate parts, a *global* part and a *local* part, with the global part coming first.
2. In the global part, each formula is either an axiom of $\mathbf{S}$, a member of Γ, or follows from earlier formulas by Modus Ponens or Necessitation.
3. In the local part, each formula is either an axiom of $\mathbf{S}$, a member of Λ, or follows from earlier formulas by Modus Ponens (but not Necessitation).

Completeness

Theorem 64

Let $\mathbf{S}$ *be an axiomatic system for any normal modal logic that is sound and strongly complete with respect to* $\vDash_{\mathsf{F}}$*. Then* $(\Gamma)\Lambda \vdash_{\mathbf{S}} \varphi$ *iff* $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi$*.*

Outline

Update Consequence

Definition 65 (Update Consequence)

Given a class of frames F, φ is an *update consequence of the global assumptions* Γ *and the local assumptions* Λ *in* F, denoted $[\Gamma]\Lambda \vDash_{\mathsf{F}} \varphi$, if for every frame $\mathfrak{F}$ in F, for every model $\mathfrak{M} = (W, R, V)$ based on $\mathfrak{F}$, for every world w in $\mathfrak{M}[\Gamma]$ at which $\mathfrak{M}[\Gamma], w \Vdash \Lambda$, we have $\mathfrak{M}[\Gamma], w \Vdash \varphi$, where $\mathfrak{M}[\Gamma] = (W', R', V')$ is defined as follows:

$$W' = \{w \in W \mid \mathfrak{M}, w \Vdash \Gamma\}$$
$$R' = R \cap (W' \times W')$$
$$V'(p) = V(p) \cap W', \text{ for all } p \in PV$$

Update Consequence

Theorem 66

For any class of frames F *and* $\Gamma \cup \Lambda \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $[\Gamma]\Lambda \vDash_{\mathsf{F}} \varphi$ *implies* $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi$.

Fact 67

There exist a class of frames F *and* $\Gamma \cup \Lambda \cup \{\varphi\} \subseteq \mathcal{L}_\Box$ *such that* $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi$ *but* $[\Gamma]\Lambda \nvDash_{\mathsf{F}} \varphi$.

Proof.

Consider $\mathfrak{F} = (\{w, u\}, \{(w, u)\})$ and $\mathsf{F} = \{\mathfrak{F}\}$. For any model $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M} \nVdash \Box\bot$. Hence, $(\Box\bot) \vDash_{\mathsf{F}} \bot$. But there exists $\mathfrak{M}$ based on $\mathfrak{F}$ such that $\mathfrak{M}[\Box\bot], u \nVdash \bot$. Hence, $[\Box\bot] \nvDash_{\mathsf{F}} \bot$. $\Box$

Outline

Semantic consequence

Definition 68 (Semantics)

A *semantics for* $\mathcal{L}$ is a pair $\sigma = (\mathsf{M}, \Vdash)$, where M is a class of models for $\mathcal{L}$ and $\Vdash \subseteq \mathsf{M} \times \mathcal{L}$ is the truth relation. We write $m \Vdash \Gamma$ if $m \Vdash \varphi$ for all $\varphi \in \Gamma$.

Definition 69 (Semantic Consequence)

Given a semantics $\sigma = (\mathsf{M}, \Vdash)$ for $\mathcal{L}$, the *semantic consequence of* σ, denoted $\vDash_\sigma$, is a subset of $\wp(\mathcal{L}) \times \mathcal{L}$ such that $\Gamma \vDash_\sigma \varphi$ iff for all $m \in \mathsf{M}$, $m \Vdash \Gamma$ implies $m \Vdash \varphi$.

Reformulating Local and Global Consequence

Given a class of frames F, the local consequence $\vDash_{\mathsf{F}}$ and the global consequence $\vDash^g_{\mathsf{F}}$ can now be redefined by $\vDash_{\sigma\mathsf{F}}$ and $\vDash_{\gamma\mathsf{F}}$, respectively, where $\sigma\mathsf{F} = (\mathsf{M}, \Vdash^l)$ and $\gamma\mathsf{F} = (g\mathsf{M}, \Vdash^g)$ are defined as follows, and $\Vdash$ is the standard satisfaction relation in modal logic.

$$\mathsf{M} = \{(\mathfrak{M}, w) \mid \mathfrak{M} \text{ is based on a frame in } \mathsf{F} \text{ and } w \in \mathfrak{M}\}$$
$$\Vdash^l = \Vdash$$
$$g\mathsf{M} = \{\mathfrak{M} \mid \mathfrak{M} \text{ is based on a frame in } \mathsf{F}\}$$
$$\mathfrak{M} \Vdash^g \varphi :\Leftrightarrow \mathfrak{M} \Vdash \varphi$$

Reformulating Kripke Consequence

Given a class of Kripke frames $\mathcal{F}$, the Kripke consequence $\vDash^k_{\mathcal{F}}$ can now be redefined by $\vDash_{\kappa\mathcal{F}}$, where $\kappa\mathcal{F} = (k\mathsf{M}, \Vdash^k)$ is defined as follows.

$$k\mathsf{M} = \{(\mathfrak{M}^u, u) \mid \mathfrak{M}^u \text{ is based on a frame in } \mathcal{F}\}$$
$$(\mathfrak{M}^u, u) \Vdash^k \varphi :\Leftrightarrow \mathfrak{M}^u \Vdash_u \varphi$$

Outline

Axiomatizing Domain Semantics

- In (Holliday and Icard III, 2017) and (Holliday and Icard III, 2018), global consequence of $\mathbf{K45}$ and $\mathbf{KD45}$ are given respectively for axiomatizing the informational consequence, where global syntactic consequence $\vdash^g_{\mathbf{S}}$ for $\mathbf{S}$ is defined as $\Gamma \vdash^g_{\mathbf{S}} \varphi$ iff $\Box\Gamma \vdash_{\mathbf{S}} \Box\varphi$.
- Which axiomatic system is the correct one for informational consequence?
- In fact, all are correct, since we have

Theorem 70

For any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Box$, $\Gamma \vDash^g_{\mathsf{S5}} \varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{S5}} \Box\varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{K45}} \Box\varphi$ *iff* $\Box\Gamma \vDash_{\mathsf{KD45}} \Box\varphi$.

- We need a more fine-grained notion of axiomatic system for global consequence.

Rules

Definition 71

Given a language $\mathcal{L}$, a *rule* in $\mathcal{L}$ is a relation $R \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ such that for every substitution σ, if $(\Gamma, \varphi) \in R$ then $(\Gamma^\sigma, \varphi^\sigma) \in R$.

- Modus Ponens and Necessitation in $\mathcal{L}_\Box$ are represented by $\{(\{\varphi, \varphi \to \psi\}, \psi) \mid \varphi, \psi \in \mathcal{L}\}$ and $\{(\{\varphi\}, \Box\varphi) \mid \varphi \in \mathcal{L}_\Box\}$, respectively.
- In the tradition of abstract algebraic logic, instead of a subset of $\wp(\mathcal{L}) \times \mathcal{L}$, a rule in $\mathcal{L}$ is defined as an element in $\wp(\mathcal{L}) \times \mathcal{L}$, so that $(\{p, p \to q\}, q)$ and $(\{q, q \to r\}, r)$ are two rules (e.g. Kracht 1999, p. 20). In our definition, they are just two applications of the same rule.

Axiomatic Systems

Definition 72 (Axiomatic Systems)

An *axiomatic system for* $\mathcal{L}$ is a triple $\mathbf{S} = (Ax_{\mathbf{S}}, Rp_{\mathbf{S}}, Ri_{\mathbf{S}})$, where $Ax_{\mathbf{S}} \subseteq \mathcal{L}$ is the set of all instances of axiom schemes of $\mathbf{S}$, $Rp_{\mathbf{S}} \cup Ri_{\mathbf{S}}$ is the set of rules of $\mathbf{S}$. Elements in $Rp_{\mathbf{S}}$ are called rules of proof of $\mathbf{S}$; elements in $Ri_{\mathbf{S}}$ are called rules of inference of $\mathbf{S}$.

We say that $\mathbf{S}'$ is an *axiomatic extension* of $\mathbf{S}$, if $Rp_{\mathbf{S}} = Rp_{\mathbf{S}'}$, $Ri_{\mathbf{S}} = Ri_{\mathbf{S}'}$, and $Ax_{\mathbf{S}} \subseteq Ax_{\mathbf{S}'}$.

Theorems of Axiomatic Systems

Definition 73

Given an axiomatic system $\mathbf{S} = (Ax_{\mathbf{S}}, Rp_{\mathbf{S}}, Ri_{\mathbf{S}})$, φ is a *theorem of* $\mathbf{S}$, if there exists a proof of φ in $\mathbf{S}$, namely, if there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for each $i \leq n$,

- either $\varphi_i \in Ax_{\mathbf{S}}$, or
- there exist $\varphi_{i1}, \ldots, \varphi_{im}$ and a rule $R \in Rp_{\mathbf{S}} \cup Ri_{\mathbf{S}}$ such that $ij < i$ for $1 \leq j \leq m$ and $(\{\varphi_{i1}, \ldots, \varphi_{im}\}, \varphi_i) \in R$, i.e. φ_i is obtained from previous formulas in the sequence by applying a rule of $\mathbf{S}$.

We denote by $Th(\mathbf{S})$ the set of all theorems of $\mathbf{S}$.

Syntactic Consequence Based on Axiomatic Systems

Definition 74 (Syntactic consequence)

Given an axiomatic system $\mathbf{S} = (Ax_{\mathbf{S}}, Rp_{\mathbf{S}}, Ri_{\mathbf{S}})$, φ is *a syntactic consequence of* Γ *in* $\mathbf{S}$, denoted $\Gamma \vdash_{\mathbf{S}} \varphi$, if there exists a derivation of φ from Γ, namely, if there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for each $i \leq n$,

- either $\varphi_i \in \Gamma$, or
- $\varphi_i \in Th(\mathbf{S})$, or
- there exist $\varphi_{i1}, \ldots, \varphi_{im}$ and a rule $R \in Ri_{\mathbf{S}}$ such that $ij < i$ for $1 \leq j \leq m$ and $(\{\varphi_{i1}, \ldots, \varphi_{im}\}, \varphi_i) \in R$, i.e. φ_i is obtained from previous formulas in the sequence by applying a *rule of inference* of $\mathbf{S}$.

Reformulating Local and Global Consequence

- For example, the axiomatic system for the minimal normal modal logic is $\mathbf{K} = (\mathrm{TAUT} \cup \{\mathrm{K}\}, \{\mathrm{RN}\}, \{\mathrm{MP}\})$, where TAUT is the set of tautologies in propositional logic and K is the axiom scheme $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
- Given an axiomatic system $\mathbf{S} = (Ax_{\mathbf{S}}, Rp_{\mathbf{S}}, Ri_{\mathbf{S}})$ for a normal modal logic, define $\mathbf{S}^g = (Ax_{\mathbf{S}^g}, Rp_{\mathbf{S}^g}, Ri_{\mathbf{S}^g})$ as follows:

$$\begin{aligned} Ax_{\mathbf{S}^g} &= Ax_{\mathbf{S}} \\ Rp_{\mathbf{S}^g} &= Rp_{\mathbf{S}} - \{\mathrm{RN}\} \\ Ri_{\mathbf{S}^g} &= Ri_{\mathbf{S}} \cup \{\mathrm{RN}\} \end{aligned}$$

- For example, $\mathbf{K}^g = (\mathrm{TAUT} \cup \{\mathrm{K}\}, \emptyset, \{\mathrm{MP}, \mathrm{RN}\})$.
- Now we have $\vdash^g_{\mathbf{S}} = \vdash_{\mathbf{S}^g}$.

The Feature of Global Consequence

- Since RN is treated as a rule of inference in $\mathbf{S}^g$, it seems that for global consequence, premises play the same role as axioms. This was reflected in (Venema, 1992), where $\vdash^*_\Lambda$ was used for the syntactic version of the global consequence for the logic with Λ the set of axioms.
- The author claimed that, compared with the local consequence $\vdash_\Lambda$, $\vdash^*_\Lambda$ can not distinguish $\Lambda_1 \vdash^*_{\Lambda_2} \varphi$ from $\Lambda_2 \vdash^*_{\Lambda_1} \varphi$, since both are equivalent to $\Lambda_1 \cup \Lambda_2 \vdash^*_{\mathrm{K}} \varphi$.
- However, this is incorrect. Unlike axioms, in global consequence uniform substitution can not be applied to premises, even though the rule of modal generalization is applicable to them. So we have $\vdash^*_{\Box p \to p} \Box q \to q$, but not $\Box p \to p \vdash^* \Box q \to q$.

One Global Consequence is Enough

- But we do have $\{\Box\psi \to \psi \mid \psi \in \mathcal{L}_\Box\} \vdash^*_{\mathrm{K}} \varphi$ iff $\vdash^*_{\mathrm{KT}} \varphi$, where $\mathrm{KT} = \{\mathrm{K}, \Box\varphi \to \varphi\}$.
- In our notation, $\vdash_{\mathbf{K}^g}$ can define $\vdash_{\mathbf{S}^g}$, for all axiomatic extensions $\mathbf{S}$ of $\mathbf{K}$.
- In this sense, for axiomatizable normal modal logics, we need only one global consequence, namely $\vdash_{\mathbf{K}^g}$.

Outline

A General Tenary Consequence

Definition 75 (General semantics)

A *general semantics* for $\mathcal{L}$ is a triple $\gamma = (\mathsf{M}, \Vdash, h)$, where $(\mathsf{M}, \Vdash)$ is a semantics for $\mathcal{L}$ and $h : \wp(\mathcal{L}) \to \wp(\mathsf{M})$ is a choice function that selects a subset of M for each $\Delta \subseteq \mathcal{L}$.

Definition 76 (General semantic consequence)

φ is a *general semantic consequence of* Λ *under the assumptions* Γ *with respect to* γ, denoted $(\Gamma)\Lambda \vDash_\gamma \varphi$, if for all $m \in h(\Gamma)$, $m \Vdash \Lambda$ implies $m \Vdash \varphi$.

Reformulating Fitting Consequence

Then Fitting consequence $(\Gamma)\Lambda \vDash_{\mathsf{F}} \varphi$ can be redefined by $(\Gamma)\Lambda \vDash_{f\gamma} \varphi$, where $f\gamma = (f\mathsf{M}, \Vdash^f, h^f)$ is defined as follows.

$$f\mathsf{M} = \{(\mathfrak{M}, w) \mid \mathfrak{M} \text{ is based on a frame in } \mathsf{F} \text{ and } w \in \mathfrak{M}\}$$

$$\Vdash^f = \Vdash$$

$$h^f(\Delta) = \{(\mathfrak{M}, w) \in \mathsf{M} \mid \mathfrak{M} \Vdash \Delta\}$$

Reformulating Local and Global Consequence

- The local consequence $\Gamma \vDash_{\mathsf{F}} \varphi$ can be redefined by $(\emptyset)\Gamma \vDash_{f\gamma} \varphi$.
- The global consequence $\Gamma \vDash^{g}_{\mathsf{F}} \varphi$ can be redefined by $(\Gamma)\emptyset \vDash_{f\gamma} \varphi$.

Reformulating Kripke Consequence

The Kripke consequence $\Gamma \vDash^k_{\mathcal{F}} \varphi$ can be redefined by $(\emptyset)\Gamma \vDash_{k\gamma} \varphi$, where $k\gamma = (k\mathsf{M}, \Vdash^k, h^k)$ is defined as follows.

$$k\mathsf{M} = \{(\mathfrak{M}^u, w) \mid \mathfrak{M}^u \text{ is based on a frame in } \mathcal{F} \text{ and } w \in \mathfrak{M}^u\}$$

$$(\mathfrak{M}^u, w) \Vdash^k \varphi :\Leftrightarrow \mathfrak{M}^u \Vdash_w \varphi$$

$$h(\Delta) = \{(\mathfrak{M}^u, u) \mid \mathfrak{M}^u \in k\mathsf{M}\}$$

Reformulating Information Consequence

The informational consequence $\Gamma \vDash_I \varphi$ can be redefined by $(\Gamma)\emptyset \vDash_{i\gamma} \varphi$, where $i\gamma = (i\mathsf{M}, \Vdash^i, h^i)$ is defined as follows.

$$i\mathsf{M} = \{(\mathfrak{D}, w, i) \mid \mathfrak{D} = (W, V) \text{ is a domain model and } w \in i \subseteq W\}$$

$$\Vdash^i = \Vdash \text{ in domain semantics}$$

$$h^i(\Delta) = \{(\mathfrak{D}, w, i) \in i\mathsf{M} \mid \mathfrak{D}, i \Vdash \Delta\}$$

Conclusions

- There is a trade-off between formal language and logical consequence. One can define one consequence by another, provided with more expressive languages.
- We probably need only one logical consequence, at least for axiomatizable modal logics, with additional axioms formulated as global assumptions based on $\mathbf{K}$.
- Different notions of validity can be defined by the same notion of validity, based on different semantics or axiomatic systems.
- Distinguishing global assumptions with local ones and respectively rules of proof with rules of inference may be instructive and beneficial to applications in natural language.
- A uniform framework for logical consequence in modal logic is possible.

Future Work

- More semantics and logical consequences in modal logic could be investigated, in particular, two-dimensional semantics.
- More modal logics could be investigated, including non-normal modal logic, hybrid logic, dynamic epistemic logic, and subset space logic.
- The consequence of the technical results in philosophy of logic could be investigated, for example, how they are related with logical pluralism.

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